Counting Colorings on Cubic Graphs

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For any $q \geq 3$, it is NP-hard to decide whether a graph is $q$-colorable
Counting Proper Colorings

• Counting is harder:

$\#P$-hard on cubic graphs for any $q \geq 3$[BDGJ99]
Approximate Counting
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It is natural to approximate the number of proper colorings
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FPRAS/FPTAS

Compute $\hat{Z}$ in $\text{poly}(G, \frac{1}{\varepsilon})$ time satisfying

$$(1 - \varepsilon)\hat{Z} \leq Z(G) \leq (1 + \varepsilon)\hat{Z}$$
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Any polynomial-factor approximation algorithm can be boosted into an FPRAS/FPTAS [JVV 1986]
Problem Setting
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FPRAS can distinguish between zero and nonzero $Z(G)$
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Assume $q \geq \Delta + 1$, 
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Exact Counting: Hard
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Assume $q \geq \Delta + 1$,

Decision: Easy

Exact Counting: Hard

Approximate Counting: ?
Uniqueness Threshold
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\[ q = \Delta \]

\{...\}
Uniqueness Threshold

\[ q = \Delta \]
\[ \left\{ \text{smaller tree} \right\} \]

\[ q = \Delta + 1 \]
\[ \left\{ \text{larger tree} \right\} \]
Uniqueness Threshold

\[ q = \Delta \]
\{ \ldots \} 

\[ q = \Delta + 1 \]
\{ \ldots \} 

The Gibbs measure is unique for \( q \geq \Delta + 1 \) [Jonasson 2002]
Two-Spin System
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In (anti-ferromagnetic) two-spin system, the uniqueness threshold corresponds to the approximability threshold.
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\[ \text{NP} \neq \text{RP} \]

**Theorem.** [Weitz06, Sly10, SS11, SST12, LLY12, LLY13] Assume \( \text{NP} \neq \text{RP} \), the partition function of anti-ferromagnetic two-spin system is approximable if and only if the Gibbs measure is unique.
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**Theorem.** [Weitz06, Sly10, SS11, SST12, LLY12, LLY13] Assume $\text{NP} \neq \text{RP}$, the partition function of anti-ferromagnetic two-spin system is approximable if and only if the Gibbs measure is unique.

Similar results in multi-spin system (Coloring)?
Counting and Sampling
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Theorem. [JVV 1986]

FPRAS $\leftrightarrow$ FPAUS
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fully-polynomial time approximate uniform sampler
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Theorem. [JVV 1986]

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fully-polynomial time approximate uniform sampler

It is natural to use Markov chains to design sampler
A Markov Chain to Sample Colorings
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Colors { on red blue green purple }
A Markov Chain to Sample Colorings

Colors \{ \textcolor{yellow}{\bullet} \textcolor{red}{\bullet} \textcolor{green}{\bullet} \textcolor{blue}{\bullet} \textcolor{purple}{\bullet} \}
A Markov Chain to Sample Colorings

Colors \{ \bullet \bullet \bullet \bullet \ \bullet \}
A Markov Chain to Sample Colorings

Colors \{ \text{ \textcolor{yellow}{\textbullet} \textcolor{red}{\textbullet} \textcolor{green}{\textbullet} \textcolor{blue}{\textbullet} \textcolor{purple}{\textbullet} } \}
A Markov Chain to Sample Colorings

Colors \{ \textcolor{yellow}{\circ}, \textcolor{red}{\circ}, \textcolor{green}{\circ}, \textcolor{blue}{\circ}, \textcolor{purple}{\circ} \}

(One-site) Glauber Dynamics
A Markov Chain to Sample Colorings

Colors \{ \text{red, blue, green, purple} \}

(One-site) Glauber Dynamics

A uniform sampler when \( q \geq \Delta + 2 \)
Analysis of the Mixing Time
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Rapid mixing of the Glauber dynamics implies FPRAS for colorings
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Relax the requirement to $q \geq \alpha \cdot \Delta + \beta$
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Ideally, $\alpha = 1$ and $\beta = 2$. Current best, $\alpha = \frac{11}{6}$ [Vigoda 1999]
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Ideally, \( \alpha = 1 \) and \( \beta = 2 \). Current best, \( \alpha = \frac{11}{6} \) [Vigoda 1999]

On cubic graphs, \( q = 5 \) \((= \Delta + 2)\) [BDGJ 1999]
\[ q = \Delta + 1 \]
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When \( q = \Delta + 1 \), then chain is no longer ergodic
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Colors \( \{ \textcolor{yellow}{\bullet} \textcolor{red}{\bullet} \textcolor{green}{\bullet} \textcolor{blue}{\bullet} \} \)
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When \( q = \Delta + 1 \), then chain is no longer ergodic.

Colors \{ yellow, red, green, blue \}

Frozen!
\[ q = \Delta + 1 \]

When \( q = \Delta + 1 \), then chain is no longer ergodic

Colors \{ \textcolor{yellow}{\circ}, \textcolor{red}{\circ}, \textcolor{green}{\circ}, \textcolor{blue}{\circ} \}

The method cannot achieve the uniqueness threshold
Recursion Based Algorithm
Recursion Based Algorithm

\[ G = \]

\[
\begin{array}{c}
  \text{\hspace{1em}} \\
  \text{\hspace{1em}} \\
  \text{\hspace{1em}} \\
  \text{\hspace{1em}} \\
  \text{\hspace{1em}} \\
\end{array}
\]
Recursion Based Algorithm

\[ G = \]

\[ \{ \text{node1, node2, node3, node4} \} \]
Recursion Based Algorithm

\[ G = \]

\[ \Pr_G [\circ = \bullet] = \frac{(1 - \Pr_{\bullet} [\otimes = \bullet])^2}{\sum \in \{\bullet, \circ, \circ, \circ\} (1 - \Pr_{\bullet} [\otimes = \circ])^2}. \]
Recursion Based Algorithm

\[
G = \begin{array}{c}
\circ \\
\quad \bullet \\
\quad \bullet \\
\quad \bullet \\
\quad \bullet \\
\end{array}
\]

\[
\text{Pr}_G [\circ = \bullet] = \frac{(1 - \text{Pr}_\otimes [\otimes = \bullet])^2}{\sum_{\bullet \in \{\bullet, \cdot, \circ, \triangle, \square\}} (1 - \text{Pr}_\otimes [\otimes = \bullet])^2}.
\]

This recursion can be generalized to arbitrary graphs

\[
x = f_d(x_{11}, \ldots, x_{1q}; x_{21}, \ldots x_{2q}; \ldots, x_{d1}, \ldots, x_{dq})
\]
Recursion Based Algorithm

\[ G = \{ \text{nodes} \} \]

Computing the marginal is equivalent to counting colorings [JVV 1986]

\[ \sum_{\bullet \in \{\text{nodes}\}} (1 - \Pr[\text{nodes}]) \]

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\[ x = f_d(x_{11}, \ldots, x_{1q}; x_{21}, \ldots x_{2q}; \ldots, x_{d1}, \ldots, x_{dq}) \]
Correlation Decay
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Faithfully evaluate the recursion requires $\Omega((\Delta q)^n)$ time
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Uniqueness Condition: the correlation between the root and boundary decays.
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Uniqueness threshold suggests $q \geq \Delta + 1$ is sufficient for CD to hold.
Sufficient Condition
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Very hard to rigorously establish the decay property
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Very hard to rigorously establish the decay property

A sufficient condition is

\[ |f_d(x) - f_d(\tilde{x})| \leq \gamma \cdot \|x - \tilde{x}\|_\infty \]

for some \( \gamma < 1 \)
Sufficient Condition

Very hard to rigorously establish the decay property

A sufficient condition is

$$|f_d(x) - f_d(\tilde{x})| \leq \gamma \cdot \|x - \tilde{x}\|_\infty$$

for some $\gamma < 1$

Based on the idea, the best bounds so far is

$q > 2.581\Delta + 1$ [LY 2013]
One Step Contraction
One Step Contraction

Establish $\gamma \leq \|\nabla f_d(x)\|_1 < 1$ for $x = (x_1, \ldots, x_d) \in D$

The domain $D$ is all the possible values of marginals.
One Step Contraction

Establish \( \gamma \leq \|\nabla f_d(x)\|_1 < 1 \) for \( x = (x_1, \ldots, x_d) \in D \)

The domain \( D \) is all the possible values of marginals.

\( D \) is too complicated to work with

Introduce an upper bound \( u_i \) for each \( x_i \)

Work with the polytope \( P := \{ x \mid 0 \leq x_i \leq u_i \} \).
Barrier
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It can be shown that for some $\xi > 0$, the one-step contraction does not hold for $q = (2 + \xi)\Delta$
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To improve, one needs to find a better relaxation
General Recursion
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$$\text{Pr}_{G,L} [c(v) = i] = \frac{\prod_{k=1}^{d} (1 - \text{Pr}_{G_v,L_k,i} [c(v_k) = i])}{\sum_{j \in L(v)} \prod_{k=1}^{d} (1 - \text{Pr}_{G_v,L_k,j} [c(v_k) = j])}$$
General Recursion

\[
\Pr_{G,L}[c(v) = i] = \frac{\prod_{k=1}^{d} (1 - \Pr_{G_{v},L_{k},i}[c(v_k) = i])}{\sum_{j \in L(v)} \prod_{k=1}^{d} (1 - \Pr_{G_{v},L_{k},j}[c(v_k) = j])}
\]

A vertex of degree \(d\) branches into \(d \times q\) subinstances
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A vertex of degree \(d\) branches into \(d \times q\) subinstances

The subinstances corresponding to the first child are marginals on the same graph
Constraints
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\[ x = f_d(x_{11}, \ldots, x_{1q}; x_{21}, \ldots x_{2q}; \ldots, x_{d1}, \ldots, x_{dq}) \]
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Analyze \( \nabla f_d \) on the hyperplane \( \sum_{i=1}^{q} x_{1i} = 1 \)
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Analyze \( \nabla f_d \) on the hyperplane \( \sum_{i=1}^{q} x_{1i} = 1 \)

Introduce the polytope \( P' := P \cap \{ x \mid \sum_{i=1}^{q} x_{1i} = 1 \} \)
Technical Issue
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In fact, we need to analyze an amortized function $\varphi \circ f_d$
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The constraint \( \sum_{i=1}^{q} x_{1i} = 1 \) breaks down in the new domain.
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In fact, we need to analyze an amortized function $\varphi \circ f_d$

The constraint $\sum_{i=1}^{q} x_{1i} = 1$ breaks down in the new domain

The use of potential function and domain restriction are not compatible
New Recursion
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We have to work with the same polytope $P$
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$$\text{Pr}_{G,L} [c(v) = i] = \frac{\prod_{k=1}^{d} (1 - \text{Pr}_{G_v,L_k,i} [c(v_k) = i])}{\sum_{j \in L(v)} \prod_{k=1}^{d} (1 - \text{Pr}_{G_v,L_k,j} [c(v_k) = j])}$$

with $\sum_{j \in [q]} \text{Pr}_{G_v,L_1,j} [c(v_1) = j] = 1$
New Recursion

We have to work with the same polytope $P$

$$
\Pr_{G,L} [c(v) = i] = \frac{\prod_{k=1}^{d} (1 - \Pr_{G_v,L_k,i} [c(v_k) = i])}{\sum_{j \in L(v)} \prod_{k=1}^{d} (1 - \Pr_{G_v,L_k,j} [c(v_k) = j])}
$$

with $\sum_{j \in [q]} \Pr_{G_v,L_1,j} [c(v_1) = j] = 1$

The constraint $\sum_{i=1}^{q} x_{1i} = 1$ is implicitly imposed if one further expand the first child with the same recursion.
New definition of one step recursion
New definition of one step recursion

The new recursion keeps more information
Cubic Graphs
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The new constraint is negligible when the degree is large
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**Theorem.**
There exists an FPTAS to compute the number of proper four-colorings on graphs with maximum degree three.
Final Remark
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More constraints?
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Other ways to establish correlation decay