Interval-like Graphs and Digraphs

Pavol Hell, Simon Fraser University

Hong Kong Theory Day, January 6, 2017

Plan

Emphasis on obstruction characterizations

- Interval graphs
- Interval bigraphs and digraphs
- Bi-arc digraphs
- Circular arc graphs

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- Interval graphs
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- Bi-arc digraphs
- Circular arc graphs

Mentioning joint work with

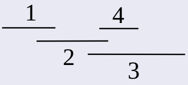
- Arash Rafiey
- Tomás Feder
- Jing Huang
- Juraj Stacho
- Mathew Francis

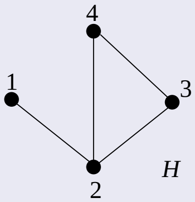
Interval graph

Vertices v can be represented by intervals I_v, so that

$$v \sim w \iff I_v \cap I_w \neq \emptyset$$

Example





Algorithms

$O(m+n)$ recognition algorithms

Booth-Lueker 1976, Korte-Mohring 1989, Habib-McConnell-Paul-Viennot 1998, Corneil-Olariu-Stewart 1998

Greedy $O(n)$ optimization algorithms

Gavril 1974, Rose-Tarjan-Lueker 1976

Interval Graphs

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Applications

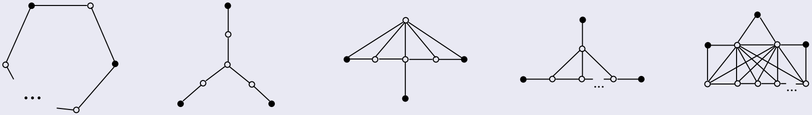
Food webs, resource allocation, genetics, etc.

Benzer 1959, Cohen 1978, Klee 1969

Interval Graphs

Lekkerkerker-Boland 1962

H is an interval graph $\iff$ H has no induced subgraph from



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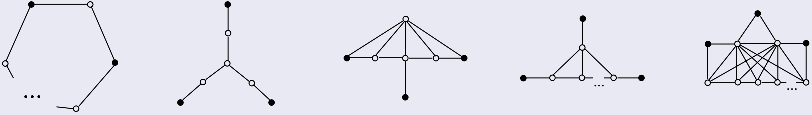
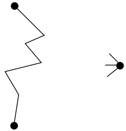


Asteroidal triple (AT)

Any two joined by a path avoiding the neighbours of the third



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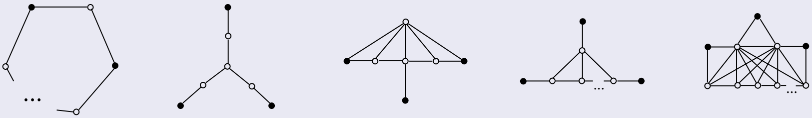

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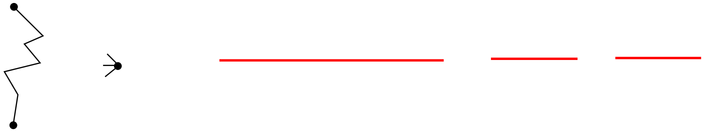

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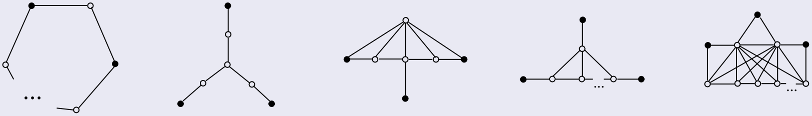

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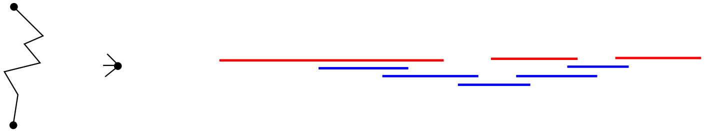

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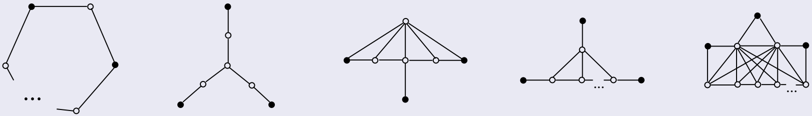
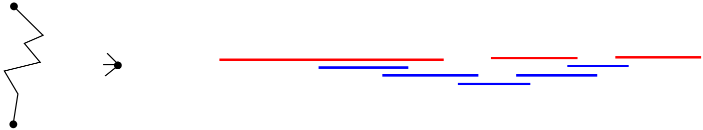
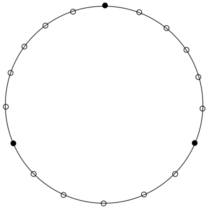


Lekkerkerker-Boland 1962

H is an interval graph $\iff$ H has no AT or induced $C_k, k \geq 4$.



A Structural Characterization



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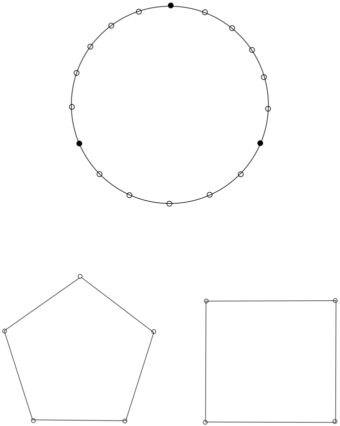

A Structural Characterization

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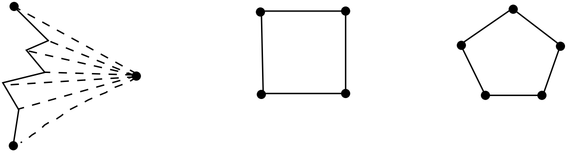
H is an interval graph $\iff$ H has no AT or induced C_4, C_5.



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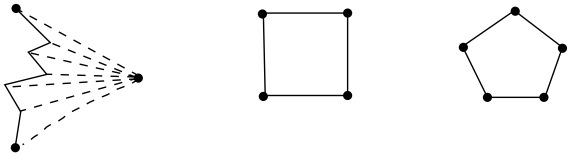

Kratsch-McConnell-Mehlhorn-Spinrad 2006

$O(m+n)$ certifying recognition algorithm (with these certificates)

An Ordering Characterization

Min-ordering

H is an interval graph

$$\iff$$

$V(H)$ can be linearly ordered by $<$ so that

$$u \sim v, u' \sim v' \text{ and } u < u', v' < v \implies u \sim v'$$

An Ordering Characterization

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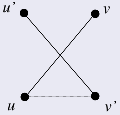
Dotted edge cannot be absent



An Ordering Characterization

Min-ordering

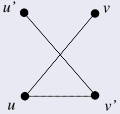
H is an interval graph

$$\Longleftrightarrow$$

$V(H)$ can be linearly ordered by $<$ so that

$$u \sim v, u' \sim v' \implies \min(u, u') \sim \min(v, v')$$

Dotted edge cannot be absent



An Ordering Characterization

Min-ordering

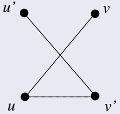
H is an interval graph

$$\iff$$

H has a *min ordering*, i.e.,
$V(H)$ can be linearly ordered by $<$ so that

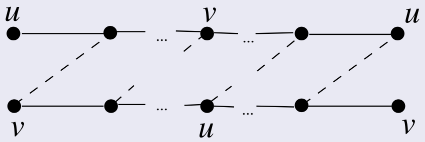
$$u \sim v, u' \sim v' \text{ and } u < u', v' < v \implies u \sim v'$$

Proof of $\implies$: order by left endpoints



An Obstruction to Min Ordering

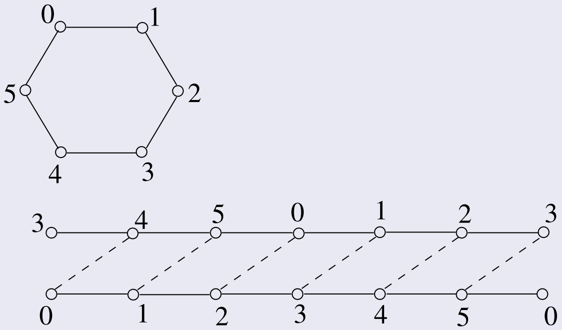
Invertible pair



Dashed line = non-edge

An Obstruction to Min Ordering

A graph with an invertible pair cannot have a min ordering



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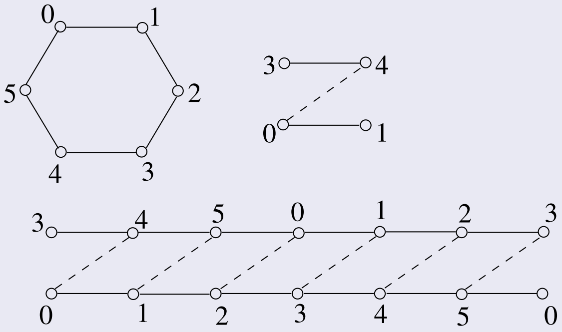

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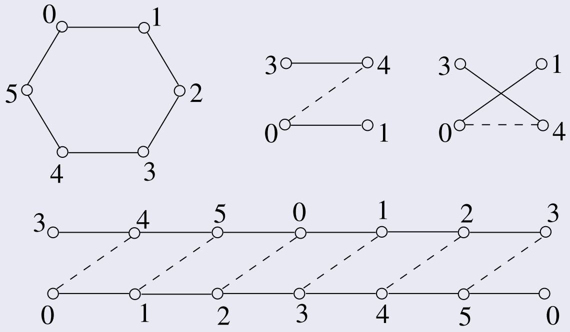
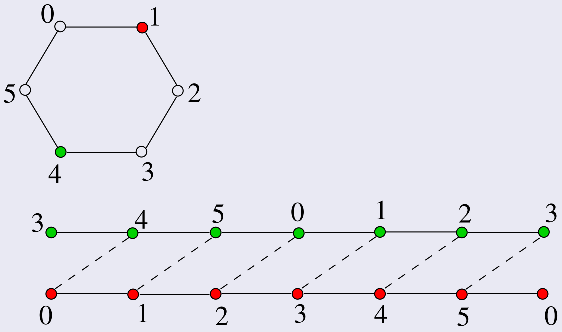


C_6 has an invertible pair



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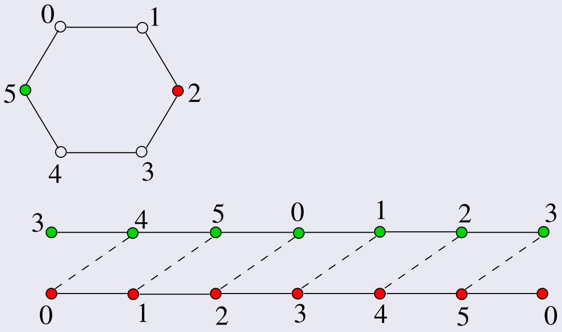

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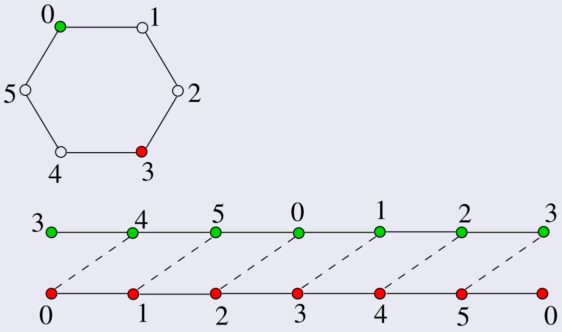

Obstruction Theorems

The following statements are equivalent

1. G is an interval graph
2. G has a min ordering
3. G has no invertible pair
4. G has no AT or induced C_4 or C_5

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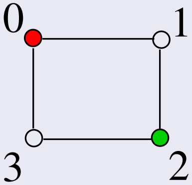
1. G is an interval graph
2. G has a min ordering
3. G has no invertible pair
4. G has no AT or induced C_4 or C_5

Shown: $1 \implies 2$, $2 \implies 3$, and $4 \implies 1$

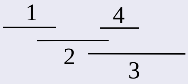
To show: $3 \implies 4$

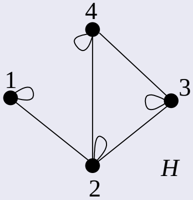
Cycles C_4, C_5 and all AT have an invertible pair



Observation

Interval graphs are reflexive (have all loops)





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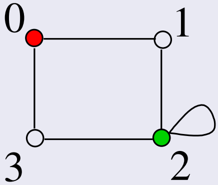

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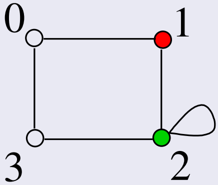

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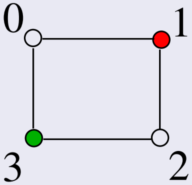

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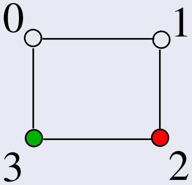

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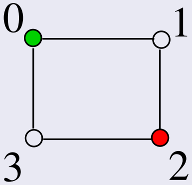

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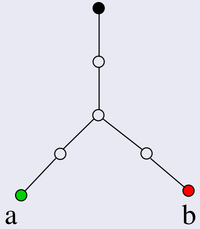

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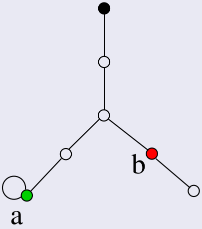

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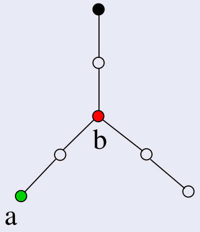

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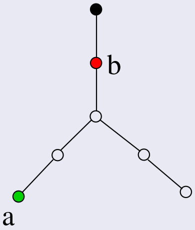

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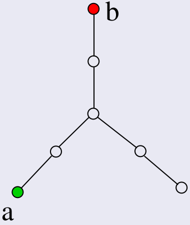

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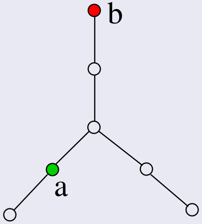

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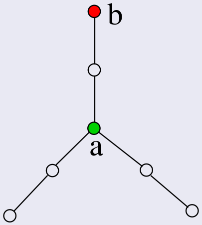

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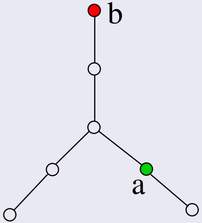

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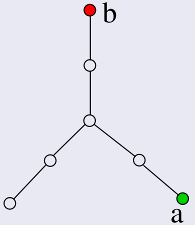

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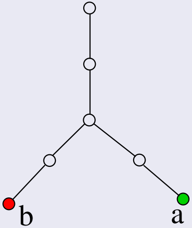
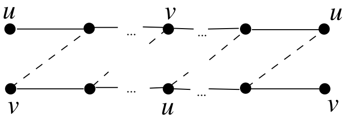


The Pulsed Obstruction Theorem

A New Characterization

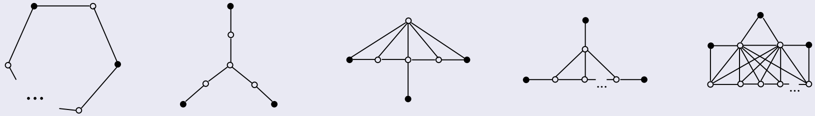
H is an interval graph $\iff$ it has no invertible pair

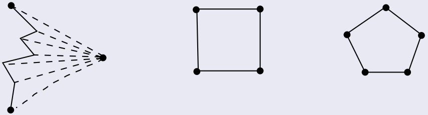


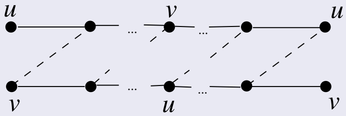
Feder+H+Huang+Rafiey 2012

Obstructions to Interval Graphs

Three related characterizations







Bigraphs

Bipartite graphs with red vs blue vertices

Bigraphs

Bipartite graphs with red vs blue vertices

Interval bigraph

Representable by real intervals I_r, J_b (for r red and b blue)

$$r \sim b \Longleftrightarrow I_r \cap J_b \neq \emptyset$$

Bipartite graphs with red vs blue vertices

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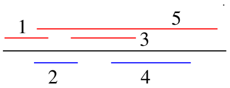

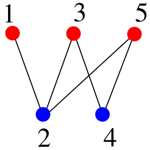

Bipartite graphs with red vs blue vertices

Interval bigraph

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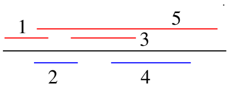

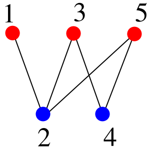

Sen-Das-Roy-West 1989

No obstruction characterizations, recognition $O(n^{15})$ Mueller 1997

Faster algorithms claimed by Rafiey 2013 and by Das 2013

Bigraphs

Min ordering of a bigraph H

A linear ordering $<$ of $V(H)$ so that

$$u \sim v, u' \sim v' \text{ and } u < u', v' < v \implies u \sim v'$$

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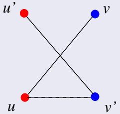


Two geometric representations

H has a min ordering $\iff$ $\overline{H}$ is a circular arc graph

Bigraphs

Two geometric representations

H has a min ordering $\iff$ $\overline{H}$ is a circular arc graph

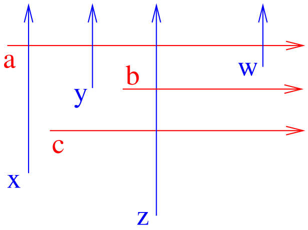
H has a min ordering $\iff$ H is a 2-directional ray graph

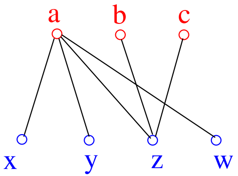
Feder, H and Huang 1999

Shrestha, Tayu, and Ueno 2010, H+Rafiey 2011

A 2DR graph

Intersection graph of a family of UP and RIGHT rays

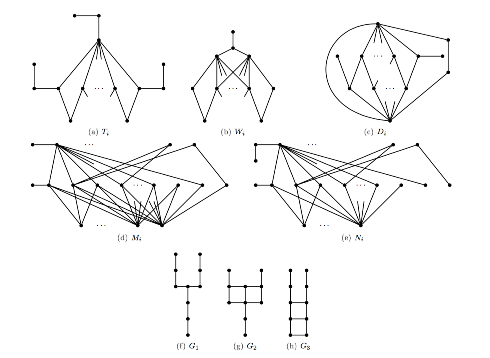




Obstruction Characterizations

A bigraph H is a 2DR graph $\iff$

- does not contain an induced cycle or any subgraph from

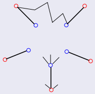


Trotter and Moore 1976

Obstruction Characterizations

A bigraph H is a 2DR graph $\iff$

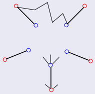
- does not contain an induced subgraph from the list
- H has no induced $C_{>4}$ and no edge-asteroid



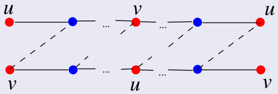
Obstruction Characterizations

A bigraph H is a 2DR graph $\iff$

- does not contain an induced subgraph from the list
- H has no induced $C_{>4}$ and no edge-asteroid



- H has no invertible pair



Trotter and Moore 1976; H and Huang 2004; H and Rafiey 2011; Shrestha, Tayu, and Ueno 2010

Similarities to interval graphs

- similar geometric representations
- similar obstructions
- similar ordering characterization

$O(n^2)$ recognition

Similarities to interval graphs

- similar geometric representations
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$O(n^2)$ recognition

Open

An $O(m+n)$ recognition algorithm?

Two Directional Ray Graphs

2DR graphs are a better analogue of interval graphs than interval bigraphs

Two Directional Ray Graphs

2DR graphs are a better analogue of interval graphs than interval bigraphs

2DR graphs are more general than interval bigraphs

- H is a 2DR graph $\iff$ $\overline{H}$ is a circular arc graph
- H is an interval bigraph $\iff$ $\overline{H}$ is a circular arc graph that can be represented without two arcs covering the circle

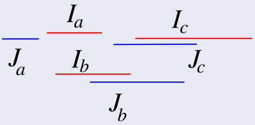
H and Huang 2004

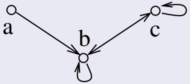
An interval digraph

Vertices can be represented by pairs of intervals I_v, J_v, so that

$$v \to w \iff I_v \cap J_w \neq \emptyset$$

Example





Sen-Das-Roy-West 1989

Digraphs

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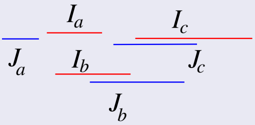

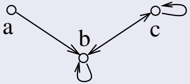

Sen-Das-Roy-West 1989

No obstruction characterization; $O(n^{15})$ recognition Mueller 1997

Faster algorithms claimed by Rafiey 2013 and by Das 2013

Digraphs

A min ordering of H

$V(H)$ can be linearly ordered by $<$ so that

$$u \to v, u' \to v' \text{ and } u < u', v' < v \implies u \to v'$$

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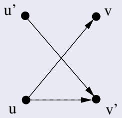


Reflexive Digraphs

A geometric representation

A reflexive digraph has has a min ordering $\iff$ it is an adjusted interval digraph

Feder+H+Huang+Rafiey 2012

A geometric representation

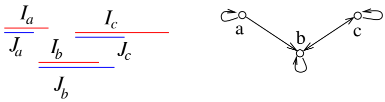
A reflexive digraph has has a min ordering $\iff$ it is an adjusted interval digraph

Feder+H+Huang+Rafiey 2012

Adjusted interval digraphs

Vertices can be represented by pairs of *adjusted* intervals I_v, J_v, so that

$$v \to w \iff I_v \cap J_w \neq \emptyset$$

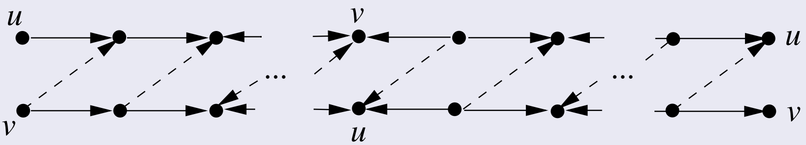


A reflexive digraph H an adjusted interval digraph if and only if

Obstruction Characterization

A reflexive digraph H an adjusted interval digraph if and only if

it has no invertible pair



Adjusted Interval diraphs

Similarities to interval graphs

- similar geometric representations
- similar obstructions
- similar ordering characterization

$O(n^4)$ recognition algorithm

Open

A more efficient recognition algorithm?

Another similarity

Dichromatic number of H

The minimum number of acyclic parts H can be partitioned into

Another similarity

Dichromatic number of H

The minimum number of acyclic parts H can be partitioned into

H is an adjusted interval digraph (without the loops)

Linear time algorithm for the dichromatic number

Hernandez-Cruz and H, 2014

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Each directed cycle in an adjusted interval digraph contains a digon

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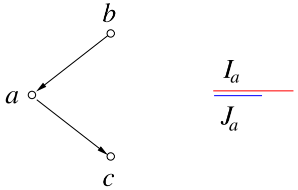

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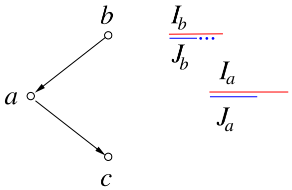

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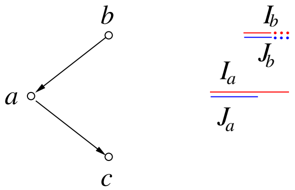

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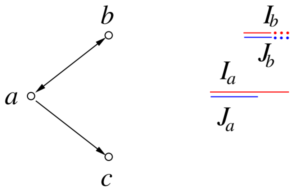
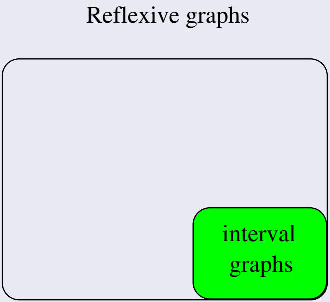
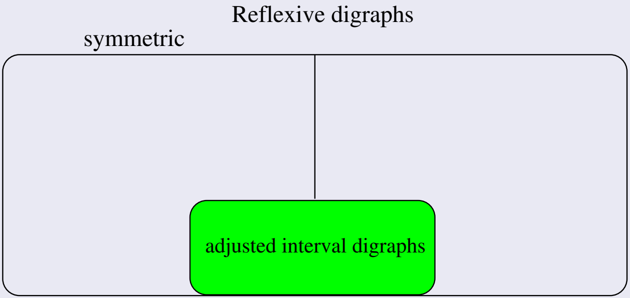


Interval-like graphs



Interval-like digraphs



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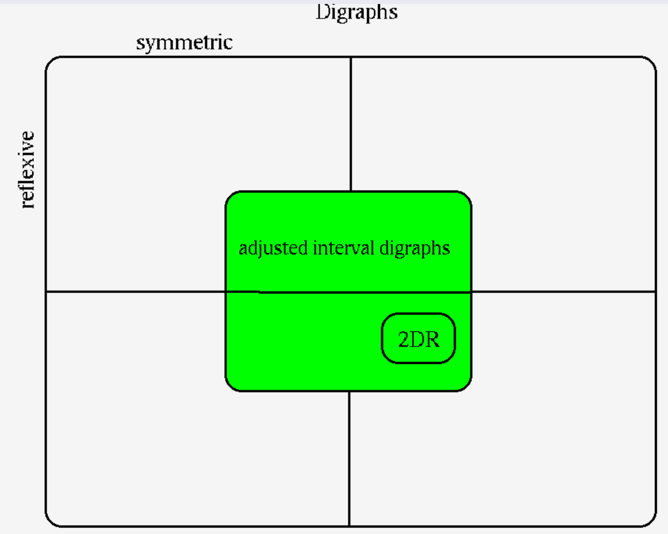

Interval-like digraphs

Interval-like digraphs

Min-orderable digraphs?

Interval-like digraphs

Min-orderable digraphs?

- Geometric representation?
- Obstruction characterization?
- Polynomial recognition algorithm?

General Digraphs

The following are equivalent

- H has a min ordering
- H is a bi-arc digraph
- H has no invertible circuit (testable in $O(n^4)$)

H+Rafiey 2016

Bi-Arc Digraphs

A bi-arc digraph H

Representable by two *consistent* families of circular arcs

$$I_v, v \in V(H), \text{ and } J_v, v \in V(H),$$

$$uv \in E(H) \iff I_u \cap J_v = \emptyset$$

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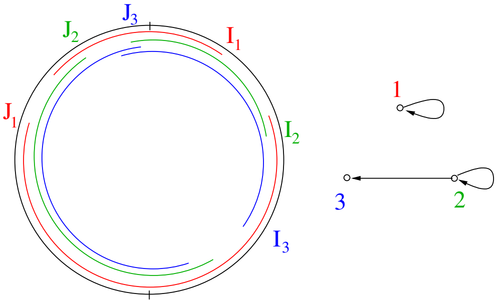

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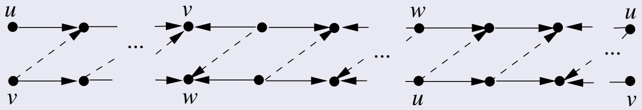
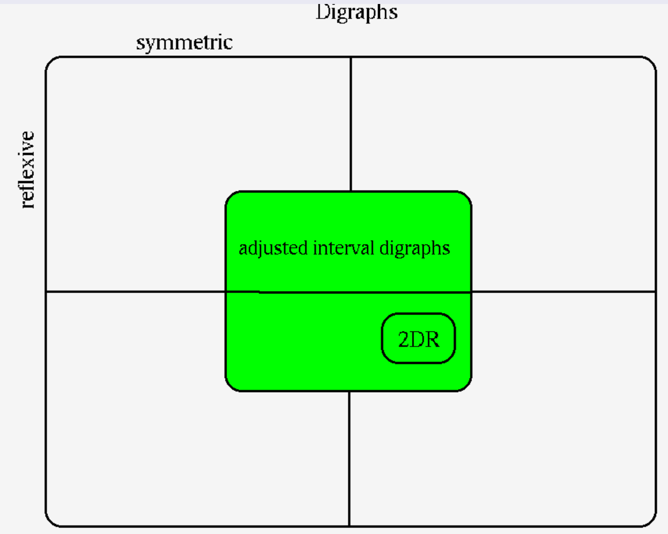
Invertible circuit



Bi-arc digraphs

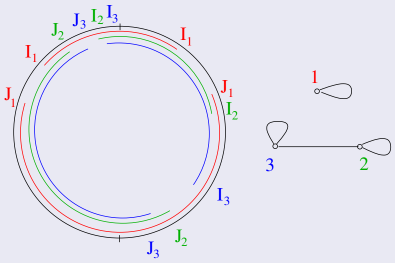


Special cases of bi-arc digraphs

Bi-Arc Digraphs

Special cases of bi-arc digraphs

- reflexive and symmetric digraph $\Longleftrightarrow$ interval graph



Same order: 1, 1, 2, 3, 2, 3

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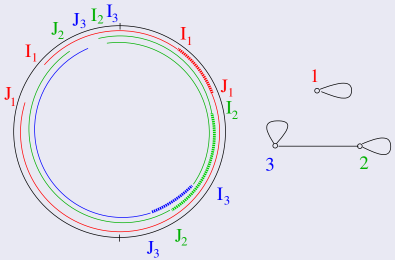

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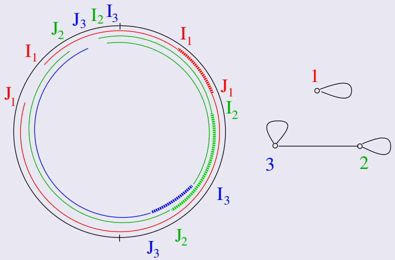

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Bi-Arc Digraphs

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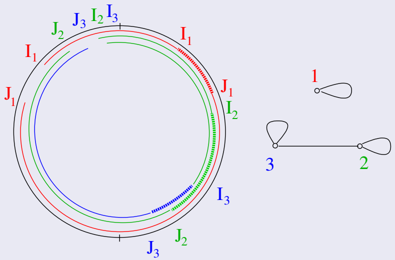

Same order: 1, 1, 2, 3, 2, 3

- reflexive digraph $\iff$ adjusted interval digraph
- bigraph $\iff$ 2DR graph

Circular arc graph

Vertices v can be represented by circular arcs I_v, so that

$$v \sim w \iff I_v \cap I_w \neq \emptyset$$

Circular Arc Graphs

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Hadwiger+Debrunner+Klee 1964

When is H is a circular arc graph?

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When is H is a circular arc graph?

Difficulties with circular arc graphs

- Helly property fails
- May have exponentially many maxcliques
- Not all perfect

Recognition algorithms

Circular Arc Graphs

Recognition algorithms

- $O(n^3)$ Tucker 1980

Circular Arc Graphs

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Certifying algorithm?

Forbidden substructure characterizations

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Circular Arc Graphs

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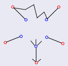
For a co-bipartite graph H

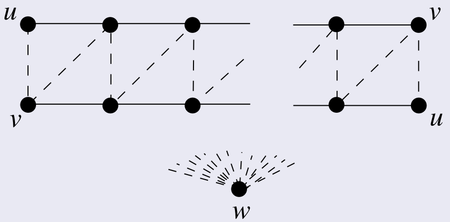
H circular arc $\iff$ $\overline{H}$ has no induced $C_{>4}$ and no edge-asteroid



A Pulsed Obstruction Characterization

A Pulsed Obstruction Characterization

An anchored invertible pair



Francis+H+Stacho 2015

A Pulsed Obstruction Characterization

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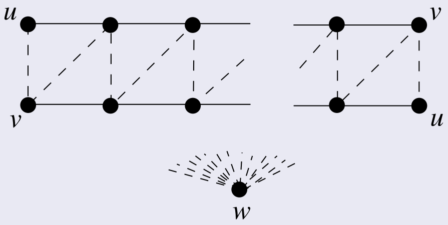
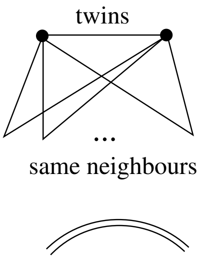
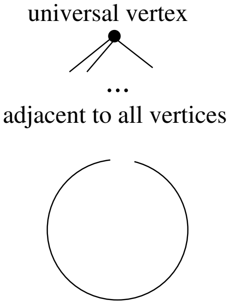


Francis+H+Stacho 2015

UNDER THE RIGHT INTERPRETATION AND ASSUMPTIONS

H has no twins and universal vertices



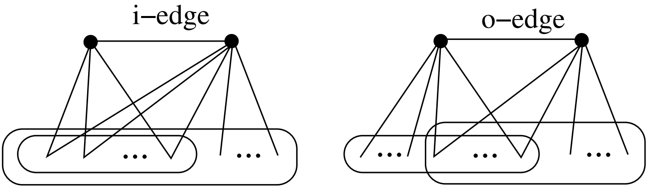


The First Twist – Standard

Each edge of H has a "type"

Type of edge uv

- Type i if $N[u] \subseteq N[v]$ ("inclusion")
- Type o if each u, v has a private neighbour ("overlap")



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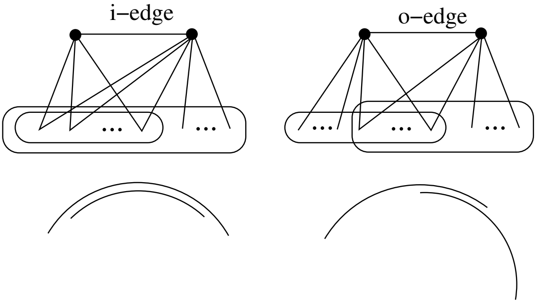

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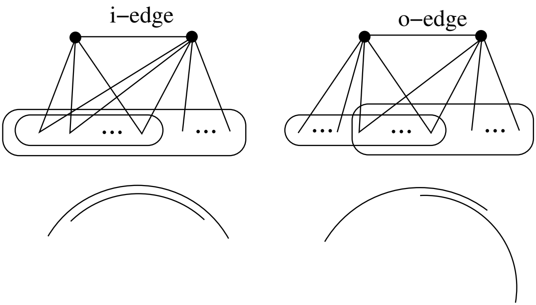

Hsu 1995

If H has a circular arc representation, then it has one corresponding to the labels

The Second Twist – New

Extend H to include "complements"

The Second Twist – New

Extend H to include "complements"

Circularly paired vertices u, v

- u and v are not adjacent
- $x \not\sim u \implies xv$ is an i-edge, and
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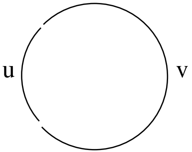

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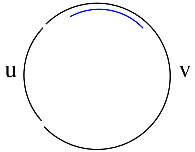

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Circular completion of H

If u is not circularly paired in H, we add a suitable new vertex $\overline{u}$ ($x \sim \overline{u} \iff xu$ is not an i-edge)

Extend H to include "complements"

Circularly paired vertices u, v

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Circular completion of H

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$(x \sim \overline{u} \iff xu$ is not an i-edge$)$

Facts

Each H has a unique circular completion H^+
H is a circular arc graph $\iff H^+$ is a circular arc graph

The Structural Characterization

Review all assumptions

- H has no twins and no universal vertices
- edges of H are labeled by their type i or o
- H is circularly complete

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- H has no twins and no universal vertices
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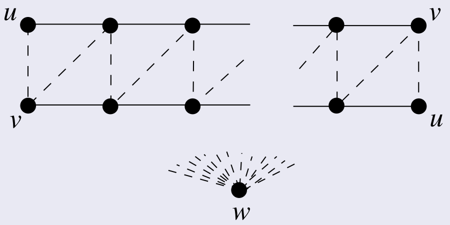
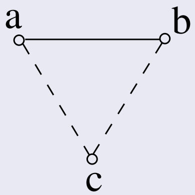
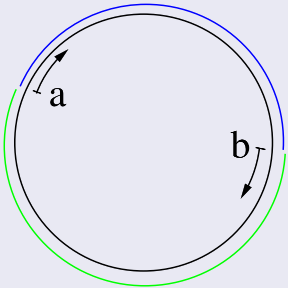
Obstruction to circular arc graphs



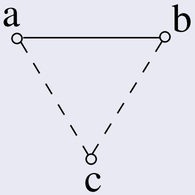
If it could be represented

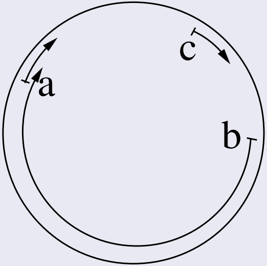




Delta Triangles

If it could be represented





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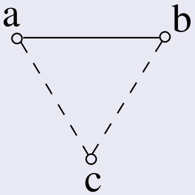

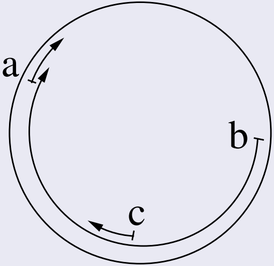

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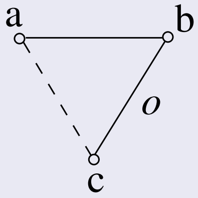

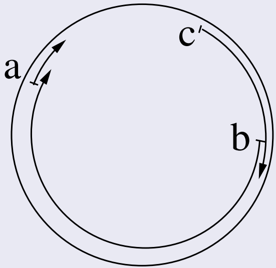

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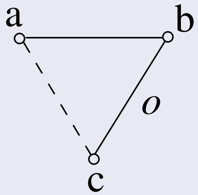

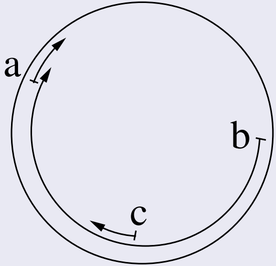

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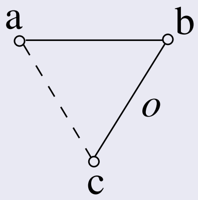

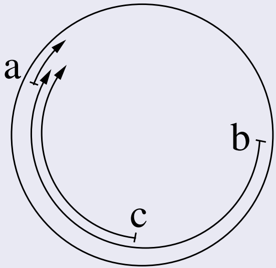

Delta Triangles

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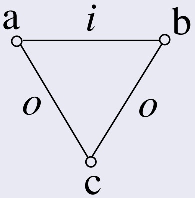

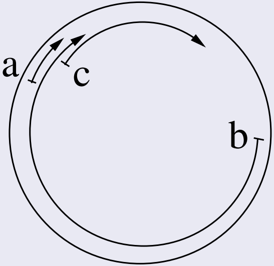

Delta Triangles

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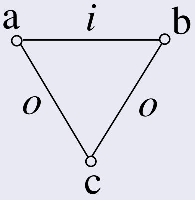
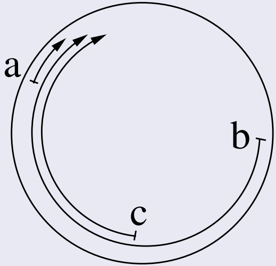
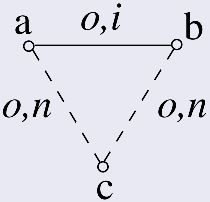
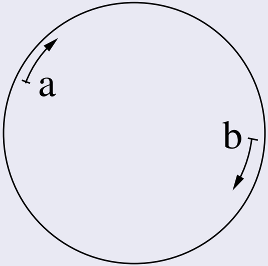




Delta Triangles

c must be on opposite side of where a meets b

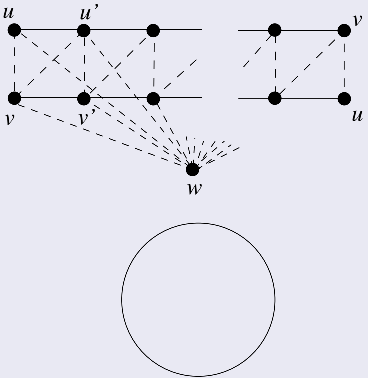




NOT ALL o

Necessity

If it could be represented



Necessity

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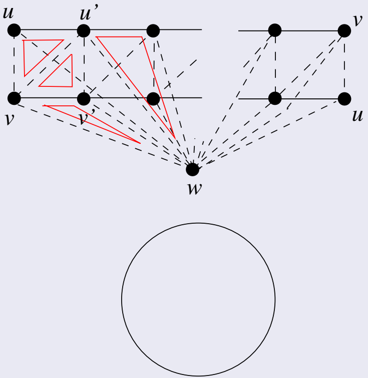

Necessity

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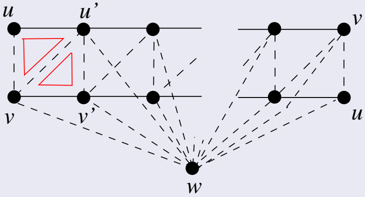
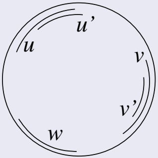
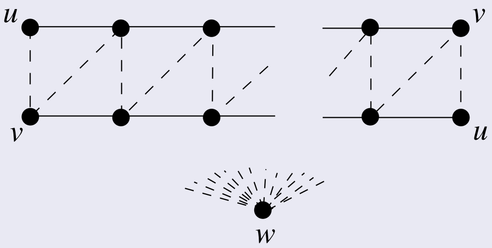


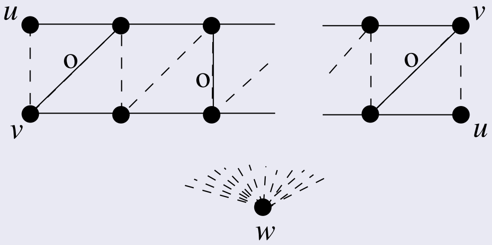


Anchored invertible pair



Dashed line = non-edge or o-edge
Each triangle with a horizontal edge is a delta triangle

Anchored invertible pair

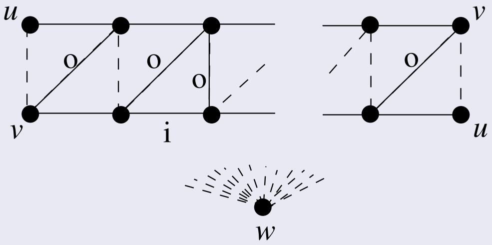


Dashed line = non-edge or o-edge

Each triangle with a horizontal edge is a delta triangle

The Structural Characterization

Anchored invertible pair



Dashed line = non-edge or o-edge
Each triangle with a horizontal edge is a delta triangle

The Structural Characterization

Assumptions

- H has no twins and no universal vertices
- edges of H are labeled by their type i or o
- H is circularly complete

Theorem

H is a circular arc graph $\iff$ it has no anchored invertible pair

Francis+H+Stacho 2015

A Certifying Algorithm

A Certifying Algorithm

Producing an anchored invertible pair

- Delete universal vertices

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Producing an anchored invertible pair

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- Delete one of each pair of twins

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- Run a standard recognition algorithm
  If a represenation is found, it is the certificate

A Certifying Algorithm

Producing an anchored invertible pair

- Delete universal vertices
- Delete one of each pair of twins
- Run a standard recognition algorithm
  If a represenation is found, it is the certificate
  If no representation is found
  - Compute the edge-labels
  - Compute the circular completion
  - Find an anchored invertible pair (via an auxiliary graph)