Interval-like Graphs and Digraphs

Pavol Hell, Simon Fraser University

Hong Kong Theory Day, January 6, 2017
Emphasis on obstruction characterizations

- Interval graphs
- Interval bigraphs and digraphs
- Bi-arc digraphs
- Circular arc graphs
Plan

**Emphasis on obstruction characterizations**

- Interval graphs
- Interval bigraphs and digraphs
- Bi-arc digraphs
- Circular arc graphs

**Mentioning joint work with**

- Arash Rafiey
- Tomás Feder
- Jing Huang
- Juraj Stacho
- Mathew Francis
Interval Graphs

Interval graph

Vertices $v$ can be represented by intervals $I_v$, so that

$$v \sim w \iff I_v \cap I_w \neq \emptyset$$

Example

```
1
  2
  3
  4
```

```
1
  2
  3
  4
```

$H$
Interval Graphs

Algorithms

$O(m + n)$ recognition algorithms


Greedy $O(n)$ optimization algorithms

Gavril 1974, Rose-Tarjan-Lueker 1976
## Algorithms

\[ O(m + n) \] recognition algorithms


Greedy \( O(n) \) optimization algorithms

Gavril 1974, Rose-Tarjan-Lueker 1976

## Applications

Food webs, resource allocation, genetics, etc.

Lekkerkerker-Boland 1962

$H$ is an interval graph $\iff H$ has no induced subgraph from

\begin{itemize}
\item \begin{tikzpicture}
  \node (a) at (0,0) [circle,fill,inner sep=1pt] {};
  \node (b) at (1,0) [circle,fill,inner sep=1pt] {};
  \node (c) at (2,0) [circle,fill,inner sep=1pt] {};
  \node (d) at (0,1) [circle,fill,inner sep=1pt] {};
  \node (e) at (1,1) [circle,fill,inner sep=1pt] {};
  \node (f) at (2,1) [circle,fill,inner sep=1pt] {};
  \draw (a) -- (b) -- (c);
  \draw (d) -- (e) -- (f);
  \draw (a) -- (d);
  \draw (b) -- (e);
  \draw (c) -- (f);
\end{tikzpicture}

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  \draw (c) -- (d);
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  \draw (b) -- (d);
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  \draw (b) -- (d);
  \draw (c) -- (d);
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\end{itemize}
Interval Graphs

Lekkerkerker-Boland 1962

$H$ is an interval graph $\iff H$ has no induced subgraph from

Any two joined by a path avoiding the neighbours of the third

Asteroidal triple (AT)
Lekkerkerker-Boland 1962

\(H\) is an interval graph \(\iff\) \(H\) has no induced subgraph from

Asteroidal triple (AT)

Any two joined by a path avoiding the neighbours of the third
Lekkerkerker-Boland 1962

$H$ is an interval graph $\iff H$ has no induced subgraph from

```
... 

```

Asteroidal triple (AT)

Any two joined by a path avoiding the neighbours of the third

```
...

```
Lekkerkerker-Boland 1962

$H$ is an interval graph $\iff H$ has no induced subgraph from

\begin{itemize}
  \item \[ \cdots \]
  \item \[ \cdots \]
  \item \[ \cdots \]
  \item \[ \cdots \]
  \item \[ \cdots \]
\end{itemize}

Lekkerkerker-Boland 1962

$H$ is an interval graph $\iff H$ has no AT or induced $C_k, k \geq 4.$
A Structural Characterization

Interval-like Graphs and Digraphs
A Structural Characterization

Lekkerkerker-Boland 1962

$H$ is an interval graph $\iff H$ has no AT or induced $C_4, C_5$. 

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Interval-like Graphs and Digraphs
A Structural Characterization

Lekkerkerker-Boland 1962

$H$ is an interval graph $\iff H$ has no AT or induced $C_4, C_5$.

Kratsch-McConnell-Mehlhorn-Spinrad 2006

$O(m + n)$ certifying recognition algorithm (with these certificates)
An Ordering Characterization

Min-ordering

$H$ is an interval graph

$\iff$

$V(H)$ can be linearly ordered by $<$ so that

$u \sim v, u' \sim v'$ and $u < u', v' < v \implies u \sim v'$
An Ordering Characterization

Min-ordering

$H$ is an interval graph

$\iff$

$V(H)$ can be linearly ordered by $<$ so that

$u \sim v$, $u' \sim v'$ and $u < u'$, $v' < v \Rightarrow u \sim v'$

Dotted edge cannot be absent

Pavol Hell, Simon Fraser University | Interval-like Graphs and Digraphs
An Ordering Characterization

Min-ordering

$H$ is an interval graph

$\iff$

$V(H)$ can be linearly ordered by $<$ so that

$u \sim v, u' \sim v' \implies \min(u, u') \sim \min(v, v')$

Dotted edge cannot be absent
An Ordering Characterization

Min-ordering

$H$ is an interval graph

$H$ has a min ordering, i.e., $V(H)$ can be linearly ordered by $<$ so that

$u \sim v, u' \sim v'$ and $u < u', v' < v \implies u \sim v'$

Proof of $\implies$ : order by left endpoints

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Interval-like Graphs and Digraphs
An Obstruction to Min Ordering

Invertible pair

Dashed line = non-edge
A graph with an invertible pair cannot have a min ordering.
A graph with an invertible pair cannot have a min ordering.

```
0 -- 1
  
5 -- 2
  
4 -- 3

3 -- 4
0 -- 1

3 4 5 0 1 2 3
0 1 2 3 4 5 0
```

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Interval-like Graphs and Digraphs
A graph with an invertible pair cannot have a min ordering.
$C_6$ has an invertible pair
$C_6$ has an invertible pair

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Interval-like Graphs and Digraphs
An Obstruction to Min Ordering

\[ C_6 \text{ has an invertible pair} \]

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Interval-like Graphs and Digraphs
The following statements are equivalent

1. G is an interval graph
2. G has a min ordering
3. G has no invertible pair
4. G has no AT or induced $C_4$ or $C_5$

Obstruction Theorems

The following statements are equivalent

1. G is an interval graph
2. G has a min ordering
3. G has no invertible pair
4. G has no AT or induced $C_4$ or $C_5$

Shown: 1 $\implies$ 2, 2 $\implies$ 3, and 4 $\implies$ 1

To show: 3 $\implies$ 4
Cycles $C_4, C_5$ and all AT have an invertible pair
Interval graphs are reflexive (have all loops)

Observation

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Interval-like Graphs and Digraphs
Cycles $C_4$, $C_5$ and all AT have an invertible pair.
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Cycles $C_4$, $C_5$ and all AT have an invertible pair
Cycles \( C_4, C_5 \) and all AT have an invertible pair

\[
\begin{align*}
& a \\
& b
\end{align*}
\]
Cycles $C_4$, $C_5$ and all AT have an invertible pair

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Interval-like Graphs and Digraphs
Cycles $C_4$, $C_5$ and all AT have an invertible pair
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Cycles $C_4$, $C_5$ and all AT have an invertible pair
A New Characterization

$H$ is an interval graph $\iff$ it has no invertible pair

Feder+H+Huang+Rafiey 2012
### Obstructions to Interval Graphs

#### Three related characterizations

<table>
<thead>
<tr>
<th>Obstruction Type 1</th>
<th>Obstruction Type 2</th>
<th>Obstruction Type 3</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.jpg" alt="Obstruction 1" /></td>
<td><img src="image2.jpg" alt="Obstruction 2" /></td>
<td><img src="image3.jpg" alt="Obstruction 3" /></td>
</tr>
</tbody>
</table>

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**Interval-like Graphs and Digraphs**
Bipartite graphs with red vs blue vertices

Interval bigraph

Representable by real intervals $I_r, J_b$ (for $r$ red and $b$ blue)

$r \sim b \iff I_r \cap J_b \neq \emptyset$

Sen-Das-Roy-West 1989

No obstruction characterizations, recognition $O(n^{15})$

Mueller 1997

Faster algorithms claimed by Rafiey 2013 and by Das 2013
Bipartite graphs with red vs blue vertices

**Interval bigraph**

Representable by real intervals $I_r, J_b$ (for $r$ red and $b$ blue)

\[ r \sim b \iff I_r \cap J_b \neq \emptyset \]
Bipartite graphs with red vs blue vertices

Interval bigraph

Representable by real intervals $I_r, J_b$ (for $r$ red and $b$ blue)

$$r \sim b \iff I_r \cap J_b \neq \emptyset$$

1 2 4 3 5

1 3 5

2 4

Sen-Das-Roy-West 1989

No obstruction characterizations, recognition $O(n^{15})$

Mueller 1997

Faster algorithms claimed by Rafiey 2013 and by Das 2013
Bigraphs

Bipartite graphs with red vs blue vertices

Interval bigraph

Representable by real intervals $I_r, J_b$ (for $r$ red and $b$ blue)

$$r \sim b \iff I_r \cap J_b \neq \emptyset$$

Sen-Das-Roy-West 1989

No obstruction characterizations, recognition $O(n^{15})$ Mueller 1997

Faster algorithms claimed by Rafiey 2013 and by Das 2013
Min ordering of a bigraph $H$

A linear ordering $<$ of $V(H)$ so that

$$u \sim v, \ u' \sim v' \text{ and } u < u', \ v' < v \implies u \sim v'$$
Min ordering of a bigraph $H$

A linear ordering $<$ of $V(H)$ so that

$$u \sim v, u' \sim v' \text{ and } u < u', v' < v \implies u \sim v'$$
Two geometric representations

$H$ has a min ordering $\iff \overline{H}$ is a circular arc graph
Two geometric representations

$H$ has a min ordering $\iff \overline{H}$ is a circular arc graph

$H$ has a min ordering $\iff H$ is a 2-directional ray graph

Feder, H and Huang 1999

Shrestha, Tayu, and Ueno 2010, H+Rafiey 2011
Two Directional Ray Graphs

A 2DR graph

Intersection graph of a family of UP and RIGHT rays

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Interval-like Graphs and Digraphs
Obstruction Characterizations

A bigraph $H$ is a 2DR graph $\iff$ does not contain an induced cycle or any subgraph from Trotter and Moore 1976.
A bigraph $H$ is a 2DR graph \iff

- does not contain an induced subgraph from the list
- $H$ has no induced $C_{>4}$ and no edge-asteroid

Trotter and Moore 1976; H and Huang 2004; H and Rafiey 2011; Shrestha, Tayu, and Ueno 2010
A bigraph $H$ is a 2DR graph $\iff$ does not contain an induced subgraph from the list

- $H$ has no induced $C_4$ and no edge-asteroid
- $H$ has no invertible pair

Trotter and Moore 1976; H and Huang 2004; H and Rafiey 2011; Shrestha, Tayu, and Ueno 2010
Two Directional Ray Graphs

Similarities to interval graphs

- similar geometric representations
- similar obstructions
- similar ordering characterization

\(O(n^2)\) recognition
Two Directional Ray Graphs

Similarities to interval graphs
- similar geometric representations
- similar obstructions
- similar ordering characterization

$O(n^2)$ recognition

Open
An $O(m + n)$ recognition algorithm?
2DR graphs are a better analogue of interval graphs than interval bigraphs.
2DR graphs are a better analogue of interval graphs than interval bigraphs

2DR graphs are more general than interval bigraphs

- $H$ is a 2DR graph $\iff \overline{H}$ is a circular arc graph
- $H$ is an interval bigraph $\iff \overline{H}$ is a circular arc graph that can be represented without two arcs covering the circle

H and Huang 2004
An interval digraph

Vertices can be represented by pairs of intervals $I_v, J_v$, so that

$$v \rightarrow w \iff I_v \cap J_w \neq \emptyset$$

Example

![Diagram showing intervals $I_a, I_b, I_c, J_a, J_b, J_c$ and the digraph $a \rightarrow b \rightarrow c$]

Sen-Das-Roy-West 1989
An interval digraph

Vertices can be represented by pairs of intervals $I_v, J_v$, so that

$$v \rightarrow w \iff I_v \cap J_w \neq \emptyset$$

Example

Sen-Das-Roy-West 1989

No obstruction characterization; $O(n^{15})$ recognition Mueller 1997

Faster algorithms claimed by Rafiey 2013 and by Das 2013
A min ordering of $H$

$V(H)$ can be linearly ordered by $<$ so that

$$u \rightarrow v, \quad u' \rightarrow v' \quad \text{and} \quad u < u', \quad v' < v \quad \implies \quad u \rightarrow v'$$
A min ordering of $H$

$V(H)$ can be linearly ordered by $<$ so that

$u \rightarrow v, u' \rightarrow v'$ and $u < u', v' < v \implies u \rightarrow v'$
Reflexive Digraphs

A geometric representation

A reflexive digraph has has a min ordering \[ \iff \] it is an adjusted interval digraph

Feder+H+Huang+Rafiey 2012

Adjusted interval digraphs

Vertices can be represented by pairs of adjusted intervals \( I_v, J_v \), so that \[ v \rightarrow w \iff I_v \cap J_w \neq \emptyset \]

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Interval-like Graphs and Digraphs
A geometric representation

A reflexive digraph has a min ordering \( \iff \) it is an adjusted interval digraph

Feder + H + Huang + Rafiey 2012
Reflexive Digraphs

A geometric representation

A reflexive digraph has has a min ordering $\iff$ it is an adjusted interval digraph

Feder++H+Huang+Rafiey 2012

Adjusted interval digraphs

Vertices can be represented by pairs of adjusted intervals $I_v, J_v$, so that

$v \rightarrow w \iff I_v \cap J_w \neq \emptyset$

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Interval-like Graphs and Digraphs
A reflexive digraph $H$ an adjusted interval digraph if and only if
A reflexive digraph $H$ is an adjusted interval digraph if and only if it has no invertible pair.
Adjusted Interval digraphs

Similarities to interval graphs
- similar geometric representations
- similar obstructions
- similar ordering characterization

$O(n^4)$ recognition algorithm

Open
A more efficient recognition algorithm?
Another similarity

Dichromatic number of $H$

The minimum number of acyclic parts $H$ can be partitioned into...
Another similarity

**Dichromatic number of** $H$

The minimum number of acyclic parts $H$ can be partitioned into

**$H$ is an adjusted interval digraph (without the loops)**

Linear time algorithm for the dichromatic number

Hernandez-Cruz and H, 2014
Another similarity

<table>
<thead>
<tr>
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Hernandez-Cruz and H, 2014

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<td>Each directed cycle in an adjusted interval digraph contains a digon</td>
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Another similarity

Dichromatic number of $H$

The minimum number of acyclic parts $H$ can be partitioned into

$H$ is an adjusted interval digraph (without the loops)

Linear time algorithm for the dichromatic number

Hernandez-Cruz and H, 2014

Directed cycles

Each directed cycle in an adjusted interval digraph contains a digon

[Diagram: Directed graph with nodes labeled a, b, c, I_a, J_a]
Another similarity

**Dichromatic number of \( H \)**

The minimum number of acyclic parts \( H \) can be partitioned into

**\( H \) is an adjusted interval digraph (without the loops)**

Linear time algorithm for the dichromatic number

Hernandez-Cruz and H, 2014

**Directed cycles**

Each directed cycle in an adjusted interval digraph contains a digon

\[
\begin{align*}
 &a \\
 &\quad \quad \quad \quad b \\
 &\quad \quad \quad \quad \quad \quad J_b \\
 &\quad \quad \quad \quad \quad \quad I_b \\
 &c \\
 &\quad \quad \quad \quad I_a \\
 &\quad \quad \quad \quad J_a
\end{align*}
\]
Another similarity

**Dichromatic number of $H$**
The minimum number of acyclic parts $H$ can be partitioned into

**$H$ is an adjusted interval digraph (without the loops)**
Linear time algorithm for the dichromatic number

Hernandez-Cruz and H, 2014

**Directed cycles**
Each directed cycle in an adjusted interval digraph contains a digon

\[ \begin{align*}
I_a &\quad J_b \\
I_b &\quad J_a
\end{align*} \]

Pavol Hell, Simon Fraser University | Interval-like Graphs and Digraphs
Another similarity

**Dichromatic number of** $H$

The minimum number of acyclic parts $H$ can be partitioned into

**$H$ is an adjusted interval digraph (without the loops)**

Linear time algorithm for the dichromatic number

Hernandez-Cruz and H, 2014

**Directed cycles**

Each directed cycle in an adjusted interval digraph contains a digon

![Diagram](image)
Interval-like graphs

Reflexive graphs

interval graphs
Interval-like digraphs

Reflexive digraphs

symmetric

adjusted interval digraphs
The World of Digraphs

Interval-like digraphs

Digraphs

symmetric

reflexive

adjusted interval digraphs

2DR
Interval-like digraphs
Interval-like digraphs

Min-orderable digraphs?
Interval-like digraphs

Min-orderable digraphs?

- Geometric representation?
- Obstruction characterization?
- Polynomial recognition algorithm?
The following are equivalent

- $H$ has a min ordering
- $H$ is a bi-arc digraph
- $H$ has no invertible circuit (testable in $O(n^4)$)

H+Rafiey 2016
A bi-arc digraph $H$

Representable by two consistent families of circular arcs

$$I_v, \ v \in V(H), \text{ and } J_v, \ v \in V(H),$$

$$uv \in E(H) \iff I_u \cap J_v = \emptyset$$
A bi-arc digraph $H$

Representable by two *consistent* families of circular arcs

$$I_v, \ v \in V(H), \text{ and } J_v, \ v \in V(H),$$

$$uv \in E(H) \iff I_u \cap J_v = \emptyset$$
The following are equivalent:

- $H$ has a min ordering
- $H$ is a bi-arc digraph
- $H$ has no invertible circuit

H+Rafiey 2016
The following are equivalent

- $H$ has a min ordering
- $H$ is a bi-arc digraph
- $H$ has no invertible circuit

H+Rafiey 2016

Invertible circuit
Bi-Arc Digraphs

Bi-arc digraphs

```
<table>
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<tr>
<th>reflexive</th>
<th>symmetric</th>
</tr>
</thead>
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<tr>
<td></td>
<td></td>
</tr>
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</table>
```

adjusted interval digraphs

2DR
Special cases of bi-arc digraphs

- Reflexive and symmetric digraphs
- Interval graph

Same order: 1, 1, 2, 3, 2, 3
Bi-Arc Digraphs

Special cases of bi-arc digraphs

- reflexive and symmetric digraph \(\iff\) interval graph

Same order: 1, 1, 2, 3, 2, 3
Special cases of bi-arc digraphs

- reflexive and symmetric digraph $\iff$ interval graph

Same order: 1, 1, 2, 3, 2, 3
Special cases of bi-arc digraphs

- reflexive and symmetric digraph $\iff$ interval graph

- reflexive digraph $\iff$ adjusted interval digraph

Same order: 1, 1, 2, 3, 2, 3
Special cases of bi-arc digraphs

- reflexive and symmetric digraph $\iff$ interval graph
  
  Same order: 1, 1, 2, 3, 2, 3

- reflexive digraph $\iff$ adjusted interval digraph

- bigraph $\iff$ 2DR graph

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Circular Arc Graphs

Circular arc graph

Vertices $v$ can be represented by circular arcs $l_v$, so that

$$v \sim w \iff l_v \cap l_w \neq \emptyset$$
Circular arc graph

Vertices \( v \) can be represented by circular arcs \( I_v \), so that

\[
 v \sim w \iff I_v \cap I_w \neq \emptyset 
\]

Hadwiger + Debrunner + Klee 1964

When is \( H \) is a circular arc graph?
Circular arc graph

Vertices $v$ can be represented by circular arcs $l_v$, so that

$$v \sim w \iff l_v \cap l_w \neq \emptyset$$

Hadwiger + Debrunner + Klee 1964

When is $H$ is a circular arc graph?

Difficulties with circular arc graphs
Circular arc graph

Vertices $v$ can be represented by circular arcs $I_v$, so that

$$v \sim w \iff I_v \cap I_w \neq \emptyset$$

Hadwiger + Debrunner + Klee 1964

When is $H$ is a circular arc graph?

Difficulties with circular arc graphs

- Helly property fails
Circular arc graphs

Vertices $v$ can be represented by circular arcs $I_v$, so that

$$v \sim w \iff I_v \cap I_w \neq \emptyset$$

Hadwiger + Debrunner + Klee 1964

When is $H$ is a circular arc graph?

Difficulties with circular arc graphs

- Helly property fails
- May have exponentially many maxcliques
Circular arc graph

Vertices $v$ can be represented by circular arcs $I_v$, so that

$$v \sim w \iff I_v \cap I_w \neq \emptyset$$

Hadwiger + Debrunner + Klee 1964

When is $H$ is a circular arc graph?

Difficulties with circular arc graphs

- Helly property fails
- May have exponentially many maxcliques
- Not all perfect
Recognition algorithms

- \( O(n^3) \) Tucker 1980
- \( O(n^2) \) Eschen+Spinrad 1993, Nussbaum 2007
- \( O(m+n) \) McConnell 2003, Kaplan+Nussbaum 2011

Certifying algorithm?

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Interval-like Graphs and Digraphs
Circular Arc Graphs

Recognition algorithms

- $O(n^3)$ Tucker 1980

Interval-like Graphs and Digraphs
Circular Arc Graphs

Recognition algorithms

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Circular Arc Graphs

Recognition algorithms

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- $O(m + n)$ McConnell 2003, Kaplan+Nussbaum 2011
## Recognition algorithms

- $O(n^3)$ Tucker 1980
- $O(n^2)$ Eschen+Spinrad 1993, Nussbaum 2007
- $O(m + n)$ McConnell 2003, Kaplan+Nussbaum 2011

Certifying algorithm?
## Forbidden substructure characterizations

- Proper CAGs
  - Tucker 1969
- Unit CAGs
  - Tucker 1969
- Co-bipartite CAGs
  - Tucker 1969, H+Huang 1999
- Helly CAGs
- Normal Helly CAGs
  - Cao-Grippo-Safe 2014
- Diamond-free CAGs, or paw-free CAGs, or \( P_4 \)-free CAGs, or claw-free chordal CAGs
  - Bonomo+Duran+Grippo+Safe 2013
- \( K_5 \)-free CAGs
  - Francis+H+Stacho 2014

For a co-bipartite graph \( H \): Circular arc \( \iff \) \( H \) has no induced \( C > 4 \) and no edge-asteroid.
Forbidden substructure characterizations

- Proper CAGs  Tucker 1969
## Forbidden substructure characterizations

- **Proper CAGs** Tucker 1969
- **Unit CAGs** Tucker 1969

For a co-bipartite graph $H$, circular arc $\iff H$ has no induced $C_4$ and no edge-asteroid.
Forbidden substructure characterizations

- **Proper CAGs** Tucker 1969
- **Unit CAGs** Tucker 1969
- **Co-bipartite CAGs** Tucker 1969, H+Huang 1999
- **Helly CAGs** Joeris+McConnell+Spinrad 2006, Lin+Szwarcfiter 2006
- **Normal Helly CAGs** Cao-Grippo-Safe 2014
- **Diamond-free CAGs**, or **paw-free CAGs**, or **P$_4$-free CAGs**, or **claw-free chordal CAGs** Bonomo+Duran+Grippo+Safe 2013
- **K$_5$-free CAGs** Francis+H+Stacho 2014

For a co-bipartite graph $H$ circular arc $\iff H$ has no induced $C_4$ and no edge-asteroid
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- **Unit CAGs** Tucker 1969
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Circular Arc Graphs

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# Circular Arc Graphs

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## For a co-bipartite graph \( H \)

\( H \) circular arc \( \iff \overline{H} \) has no induced \( C_4 \) and no edge-asteroid
An anchored invertible pair

Francis+H+Stacho 2015
An anchored invertible pair

UNDER THE RIGHT INTERPRETATION AND ASSUMPTIONS

Francis+H+Stacho 2015
\( \mathcal{H} \) has no twins and universal vertices

- **Twins**
  - Same neighbours

- **Universal vertex**
  - Adjacent to all vertices
Each edge of $H$ has a "type"

**Type of edge $uv$**

- Type $i$ if $N[u] \subseteq N[v]$ ("inclusion")
- Type $o$ if each $u, v$ has a private neighbour ("overlap")
If \( H \) has a circular arc representation, then it has one corresponding to the labels.
If $H$ has a circular arc representation, then it has one corresponding to the labels.

Hsu 1995
Extend $H$ to include "complements"
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**Circularly paired vertices $u, v$**

- $u$ and $v$ are not adjacent
- $x \not\sim u \implies xv$ is an i-edge, and
- $x \not\sim v \implies xu$ is an i-edge
Extend $H$ to include "complements"

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**Circular completion of $H$**
If $u$ is not circularly paired in $H$, we add a suitable new vertex $\overline{u}$
$(x \sim \overline{u} \iff xu$ is not an i-edge)
Extend $H$ to include "complements"

Circularly paired vertices $u, v$
- $u$ and $v$ are not adjacent
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Circular completion of $H$
If $u$ is not circularly paired in $H$, we add a suitable new vertex $\overline{u}$
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Facts
Each $H$ has a unique circular completion $H^+$
$H$ is a circular arc graph $\iff H^+$ is a circular arc graph
Review all assumptions

- $H$ has no twins and no universal vertices
- edges of $H$ are labeled by their type $i$ or $o$
- $H$ is circularly complete
Review all assumptions

- $H$ has no twins and no universal vertices
- edges of $H$ are labeled by their type $i$ or $o$
- $H$ is circularly complete

Obstruction to circular arc graphs

- Two graphs showing the obstruction
- Vertices $u$, $v$, and $w$
If it could be represented

\[\begin{align*}
  &a \quad b \\
  \quad \quad c
\end{align*}\]
Delta Triangles

If it could be represented

\[ \text{Diagram: } a \rightarrow b, \quad b \rightarrow c, \quad c \rightarrow a \]
Delta Triangles

If it could be represented

Diagram with vertices a, b, and c arranged in a triangle on the left, and a directed cycle on the right with vertices a, b, and c.
If it could be represented
Delta Triangles

If it could be represented

```
a b  o
     |
      |
     c
```

```
a a
 b
 c
```

Pavol Hell, Simon Fraser University
Interval-like Graphs and Digraphs
If it could be represented
Delta Triangles

If it could be represented

Diagram showing a triangle with vertices labeled a, i, b, and a circle labeled a, c, b.
If it could be represented

\begin{align*}
    a & \\
    o & i \\
    c & b \\
    o & \quad \quad o \\
    c & \\
    b & \\
\end{align*}

Pavol Hell, Simon Fraser University

Interval-like Graphs and Digraphs
Delta Triangles

\[ c \text{ must be on opposite side of where } a \text{ meets } b \]

\[ \text{NOT ALL } o \]
Necessity

If it could be represented
Necessity

If it could be represented

*Necessity*
Necessity

If it could be represented
The Structural Characterization

Anchored invertible pair

Dashed line = non-edge or o-edge
Each triangle with a horizontal edge is a delta triangle
The Structural Characterization

Anchored invertible pair

Dashed line = non-edge or o-edge
Each triangle with a horizontal edge is a delta triangle
Anchored invertible pair

Dashed line = non-edge or o-edge
Each triangle with a horizontal edge is a delta triangle
The Structural Characterization

Assumptions

- $H$ has no twins and no universal vertices
- edges of $H$ are labeled by their type i or o
- $H$ is circularly complete

Theorem

$H$ is a circular arc graph $\iff$ it has no anchored invertible pair

Francis+H+Stacho 2015
A Certifying Algorithm

Producing an anchored invertible pair

Delete universal vertices
Delete one of each pair of twins
Run a standard recognition algorithm
If a representation is found, it is the certificate
If no representation is found
Compute the edge-labels
Compute the circular completion
Find an anchored invertible pair (via an auxiliary graph)
A Certifying Algorithm

Producing an anchored invertible pair

- Delete universal vertices
A Certifying Algorithm

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A Certifying Algorithm

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A Certifying Algorithm

Producing an anchored invertible pair

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- Run a standard recognition algorithm
  - If a representation is found, it is the certificate
  - If no representation is found
    - Compute the edge-labels
    - Compute the circular completion
    - Find an anchored invertible pair (via an auxiliary graph)